

# The Rise of Solitons in Sine-Gordon Field Theory: From Jacobi Amplitude to Gudermannian Function

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## Abstract

We show how the famous soliton solution of the classical sine-Gordon field theory in  $(1+1)$ -dimensions may be obtained as a particular case of a solution expressed in terms of the Jacobi amplitude, which is the inverse function of the incomplete elliptic integral of the first kind.

**Keywords:** Solitons, Sine-Gordon Field Theory, Elliptic Integrals, Jacobi Amplitude

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## I. INTRODUCTION

The sine-Gordon field theory and the associated massive Thirring model [1] are some of the best studied quantum field theories. In view of its connections to other important physical models, some of which in principle admit actual realizations in nature [2, 3], a huge mass of important exact results have been obtained for this fascinating integrable system [4–7]. However, no less fascinating are the remarkable mathematical and physical properties of its soliton (or “solitary wave”) solutions which have contributed, along the last decades, to turn the physics of solitons into a very active research topic.

In this work we present a simple and yet appealing step-by-step derivation of a more general solution for the classical sine-Gordon field theory in (1+1)-dimensions in terms of a special kind of elliptic function, namely the Jacobi amplitude, which has the famous sine-Gordon soliton solution as a particular case. Despite the fact that the connection between solitons and Jacobi elliptic functions has already been explored in [8], we believe this work comes to shed more light on this interesting subject, helping to fill in a gap existing in the corresponding specialized literature.

## II. AN ALTERNATIVE PATHWAY TO SOLITONS IN SINE-GORDON FIELD THEORY

### A. The Jacobi amplitude function

We start by considering the following theory describing a real scalar field in (1+1)-dimensions ( $\phi \equiv \phi(x, t)$ ),

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi), \quad (1)$$

where the potential term is given by

$$V(\phi) = 2\alpha \cos(\beta\phi) + 2\gamma. \quad (2)$$

The above Lagrangian gives rise, through the Euler-Lagrange equation,  $\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) = \frac{\partial \mathcal{L}}{\partial \phi}$ , to the following field equation

$$\partial_\mu \partial^\mu \phi \equiv \square \phi \equiv \left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \phi = -\frac{\partial V(\phi)}{\partial \phi}. \quad (3)$$

Notice that since Equation (3) is invariant under Lorentz transformations ( $x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu$ )[9], its solutions may be obtained through the solutions of the corresponding equation for the static case ( $\phi \equiv \phi(x)$ ) by a simple Lorentz boost, namely  $x - x_0 \rightarrow (x - x_0 - vt)/\sqrt{1 - (v^2/c^2)}$ , for arbitrary  $v$  ( $|v| < c \approx 3 \times 10^8$  m/s)[10, 11]. Thus, in what follows, we will focus on the solutions of the equation

$$\frac{d^2\phi}{dx^2} = \frac{dV}{d\phi}. \quad (4)$$

Indeed, by multiplying the above equation by  $d\phi/dx$  we obtain

$$\frac{d\phi}{dx} \frac{d^2\phi}{dx^2} = \frac{d\phi}{dx} \frac{dV}{d\phi} \Rightarrow \frac{d}{dx} \left[ \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 \right] = \frac{dV}{dx}, \quad (5)$$

which, after an integration with respect to  $x$  and some algebra, may be rewritten as

$$dx' = \pm \frac{d\phi'}{\sqrt{2V(\phi')}}. \quad (6)$$

By integrating both sides of the above equation, from  $x' = x_0$  to  $x' = x$  ( $\phi' = \phi(x_0)$  to  $\phi' = \phi(x)$ ), we get

$$x - x_0 = \pm \int_{\phi(x_0)}^{\phi(x)} \frac{d\phi'}{\sqrt{2V(\phi')}}. \quad (7)$$

In order to compute the above integral, we must firstly notice that the potential, shown in Equation (2), may be rewritten as

$$V(\phi') = 2(\alpha + \gamma) \left[ 1 - \frac{2\alpha}{\alpha + \gamma} \sin^2 \left( \frac{\beta\phi'}{2} \right) \right]. \quad (8)$$

Thus, by making the change of variables  $\phi' \rightarrow \theta' = \frac{\beta}{2}\phi'$ , defining  $k^2 = \frac{2\alpha}{\alpha + \gamma}$  and choosing  $x_0$  such that  $\phi(x_0) = 0 \Rightarrow \theta_0 = 0$ , we are left with

$$x - x_0 = \pm \frac{k}{\beta\sqrt{2\alpha}} \int_0^\theta \frac{d\theta'}{\sqrt{1 - k^2 \sin^2 \theta'}}. \quad (9)$$

The integral appearing in Equation (9) is called an incomplete elliptic integral of the first kind,  $F(\theta, k)$ , whereas  $k$  is called the elliptic modulus or eccentricity. The upper limit,  $\theta$ , of this integral may be written in terms of the *Jacobi amplitude* (the inverse function of the incomplete elliptic integral of the first kind) as [12, 13].

$$\theta = \pm F^{-1} \left( \frac{\beta\sqrt{2\alpha}}{k} (x - x_0), k \right) \equiv \pm \text{am} \left( \frac{\beta\sqrt{2\alpha}}{k} (x - x_0), k \right). \quad (10)$$

Notice that, from the above definition, we have  $F(\text{am}(x, k), k) = x$ .

The solution of Equation (4) may be, finally, written as

$$\phi(x) = \pm \frac{2}{\beta} \text{am} \left( \frac{\beta \sqrt{2\alpha}}{k} (x - x_0), k \right). \quad (11)$$

Hence, from the above equation, we may notice that

$$\phi(x_0) = \pm \frac{2}{\beta} \text{am}(0, k) = 0, \quad (12)$$

as it should.

### B. The case $k = 1$ : The Gudermannian function and the soliton solution of sine-Gordon equation

From the definition  $k^2 = 2\alpha/(\alpha + \gamma)$  we may obviously see that when  $\gamma = \alpha$  we have  $k = 1$ . Hence, the solution for Equation (4) with the potential given by

$$V(\phi) = 2\alpha[1 + \cos(\beta\phi)], \quad (13)$$

may be obtained as a special case of the solution presented in Equation (11). Indeed, since  $\text{am}(x, 1) = \text{gd } x \equiv 2 \arctan(e^x) - \pi/2$ , where  $\text{gd } x$  is called the *Gudermannian function* (a special function which relates the circular functions to the hyperbolic ones without using complex numbers, named after Christoph Gudermann (1798 - 1852)), we are left with

$$\begin{aligned} \phi(x) &= \pm \frac{2}{\beta} \text{am} \left( \beta \sqrt{2\alpha} (x - x_0), 1 \right) = \pm \frac{2}{\beta} \text{gd} \left( \beta \sqrt{2\alpha} (x - x_0) \right) \\ &\equiv \pm \frac{4}{\beta} \arctan \left[ \exp \left( \beta \sqrt{2\alpha} (x - x_0) \right) \right] \mp \frac{\pi}{\beta}. \end{aligned} \quad (14)$$

Last but not least, we must notice that by substituting the Equation (14) into Equation (3) and making the change (Lorentz boost)  $x - x_0 \rightarrow (x - x_0 - vt)/\sqrt{1 - (v^2/c^2)}$ , we obtain the famous sine-Gordon field equation, namely

$$\square \phi_S + 2\alpha \beta \sin \beta \phi_S = 0, \quad (15)$$

where  $\phi_S \equiv \phi_S(x, t)$  is the no less famous soliton/anti-soliton solution [10, 11], given by

$$\phi_S(x, t) = \pm \frac{4}{\beta} \arctan \left[ \exp \left( \beta \sqrt{2\alpha} \frac{x - x_0 - vt}{\sqrt{1 - (v^2/c^2)}} \right) \right]. \quad (16)$$

This result allows us to characterize the Lorentz boosted, and shifted by  $\pi/\beta$ , version of the solution in terms of the Jacobi amplitude shown in Equation (11), namely

$$\phi(x, t) = \pm \frac{2}{\beta} \operatorname{am} \left( \frac{\beta \sqrt{2\alpha}}{k} \frac{x - x_0 - vt}{\sqrt{1 - (v^2/c^2)}}, k \right) \pm \frac{\pi}{\beta}, \quad (17)$$

as a generalization of the sine-Gordon soliton/anti-soliton solution for  $k \neq 1$ .

### III. CONCLUDING REMARKS

We would like to make a few comments about the soliton solution, shown in Equation (16), and its generalized version, shown in Equation (17). Firstly, we may notice by comparing the Figures 1 and 2 how different are these solutions, where we would like to highlight the doubly periodic behaviour of the Jacobi amplitude solution.

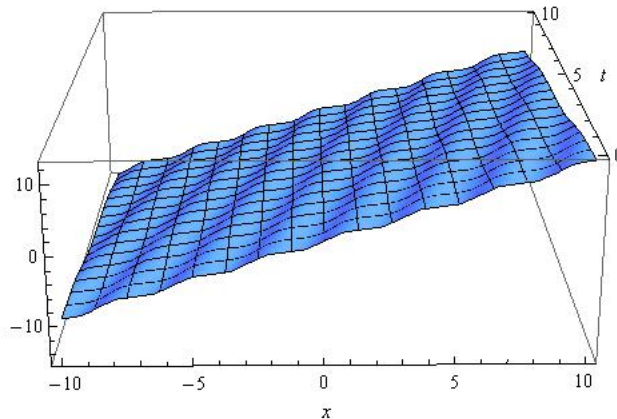


FIG. 1: The Jacobi amplitude solution given by Equation (17) with  $\alpha = 0.50$ ,  $\beta = 2.00$ ,  $x_0 = 0$ ,  $k = 0.99$  and  $v = 0.50c$

Finally, let us observe that, as remarked in [10], this soliton solution, though arising in a classical field theory, looks very much like a classical particle since its energy density is localized at a point ( $x = x_0$ ) and its total energy for a static field configuration ( $\phi_S \equiv \phi_S(x)$ ), namely

$$E(\phi_S) = \int_{-\infty}^{\infty} dx \left[ \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 + V(\phi) \right] = \frac{8\sqrt{2\alpha}}{\beta}, \quad (18)$$

is finite, just as we should expect.

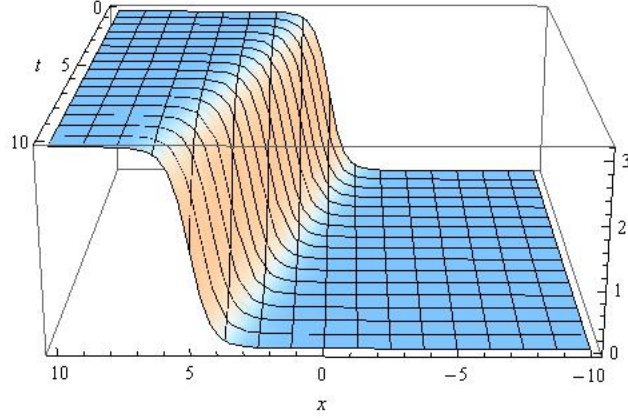


FIG. 2: The soliton solution given by Equation (16) with  $\alpha = 0.50$ ,  $\beta = 2.00$ ,  $x_0 = 0$ ,  $k = 1.00$  and  $v = 0.50c$

### Acknowledgements

This work has been supported by University of Alberta's Li Ka Shing Applied Virology Institute and CNPq, Conselho Nacional de Desenvolvimento Científico e Tecnológico - Brasil.

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- [1] Coleman, S. (1975) *Quantum Sine-Gordon Equation as the Massive Thirring Model*. Phys. Rev. D, 11, 2088-2097. <http://dx.doi.org/10.1103/PhysRevD.11.2088>
  - [2] Kosterlitz, J.M. (1974) *The Critical Properties of the Two-Dimensional XY Model*. J. Phys. C: Solid State Phys., 7, 1046-1060. <http://dx.doi.org/10.1088/0022-3719/7/6/005>
  - [3] Samuel, S. (1978) *Grand Partition Function in Field Theory with Applications to Sine-Gordon Field Theory*. Phys. Rev. D, 18, 1916-1932. <http://dx.doi.org/10.1103/PhysRevD.18.1916>
  - [4] Dauxois, T. and Peyrard, M. (2006) *Physics of Solitons*. Cambridge University Press, New York.
  - [5] Mondaini, L. and Marino, E.C. (2005) *Sine-Gordon/Coulomb Gas Soliton Correlation Functions and an Exact Evaluation of the Kosterlitz-Thouless Critical Exponent*. J. Stat. Phys., 118, 767-779. <http://dx.doi.org/10.1007/s10955-004-8828-y>
  - [6] Mondaini, L., Marino, E.C. and Schmidt, A.A. (2009) *Vanishing Conductivity of Quantum Solitons in Polyacetylene*. J. Phys. A: Math. Theor., 42, 055401. <http://dx.doi.org/10.1088/1751-8113/42/5/055401>

- [7] Mondaini, L. (2012) *Thermal Soliton Correlation Functions in Theories with a  $Z(N)$  Symmetry*. J. Mod. Phys., 3, 1776-1780. <http://dx.doi.org/10.4236/jmp.2012.311221>
- [8] Cervero, J.M. (1986) *Unveiling the Solitons Mystery: The Jacobi Elliptic Functions*. Am. J. Phys., 54, 35-38. <http://dx.doi.org/10.1119/1.14767>
- [9] Mondaini, L. (2012) *Obtaining a Closed-form Representation for the Dual Bosonic Thermal Green Function by Using Methods of Integration on the Complex Plane*. Rev. Bras. Ens. Fis., 34, 3305. <http://dx.doi.org/10.1590/S1806-11172012000300005>
- [10] Jackiw, R. (1977) *Quantum Meaning of Classical Field Theory*. Rev. Mod. Phys., 49, 681-706. <http://dx.doi.org/10.1103/RevModPhys.49.681>
- [11] Rajaraman, R. (1987) *Solitons and Instantons: An Introduction to Solitons and Instantons in Quantum Field Theory*. Elsevier, Amsterdam.
- [12] Gradshteyn, I.S. and Ryzhik, I.M. (2000) *Table of Integrals, Series, and Products*. Academic Press, San Diego.
- [13] Weisstein, E.W. *Jacobi Amplitude*. MathWorld – A Wolfram Web Resource. <http://mathworld.wolfram.com/JacobiAmplitude.html>